Kleene Algebra
"Arithmetic" Operators

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Outline

- Algebra of choice $(+)$, sequencing $(\cdot)$ and iteration $(^*)$
- Name "Kleene algebra" is a tribute to S. C. Kleene
- "Algebra of regular events"
- Lots of other interpretations.
- First example of "fixed points" and "fixed point induction".

"Arithmetic" Axioms

$$(x+y)+z = x+(y+z) \quad ,$$

$$x+y = y+x \quad ,$$

$$x+0 = x = 0+x \quad ,$$

$$x\cdot(y\cdot z) = (x\cdot y)\cdot z \quad ,$$

$$x \cdot (y+z) \;=\; (x\cdot y) + (x\cdot z) \quad ,$$

$$(y+z) \cdot x \;=\; (y\cdot x) + (z\cdot x) \quad ,$$

$$x\cdot 0 \;=\; 0 \;=\; 0\cdot x \quad ,$$

$$1\cdot x = x = x\cdot 1 \quad .$$

Overloading of "$+$" and "$\cdot$" is intended to suggest an analogy with arithmetic. But, be careful!!

Axioms — Ordering

Idempotency

$$x+x = x$$

Ordering

$$x \leq y \;\equiv\; x+y = y \quad .$$

Informal Coursework

Suppose R is a binary relation and $\oplus$ is a binary operator such that

$$x\,R\,y \quad\equiv\quad x\oplus y = y \quad .$$

Prove the following:

$$R \;\text{ is reflexive } \equiv \;\oplus\text{ is idempotent} \quad ,$$

$$R \;\text{ is transitive } \equiv \;\oplus\text{ is associative} \quad .$$

$$R \text{ is antisymmetric } \Leftarrow \;\oplus\text{ is symmetric} \quad .$$

Informal Coursework (Continued)

Show that multiplication and addition in a Kleene algebra are both monotonic.

Interpretations

	carrier	$+$	$\cdot$	0	1	$\leq$
Languages	sets of words	$\cup$	$\cdot$	ϕ	$\{\varepsilon\}$	$\subseteq$
Programming	binary relations	$\cup$	$\circ$	ϕ	id	$\subseteq$
Reachability	booleans	$\vee$	$\wedge$	false	true	$\Rightarrow$
Shortest paths	nonnegative reals	min	$+$	∞	0	$\geq$
Bottlenecks	nonnegative reals	max	min	0	∞	$\leq$